Registered School: Leota Middle School (WOODINVILLE WA)

Division **M**

Mathematical Olympiads

November 13, 2019

For Elementary & Middle Schools

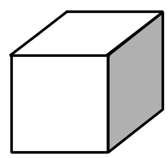
Contest **1**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

1A The prime factorization of $2020 = 2^2 \times 5 \times 101$. Find the least value of positive integer N so that $2020 \times N$ is a perfect square.

1B The six faces of a cube are to be colored so that no two faces with a common edge are the same color. What is the fewest number of different colors needed?



1C One number is selected randomly from each of the sets $\{2, 4, 6, 8, 10\}$ and $\{1, 3, 5, 7, 9\}$. Find the probability that the sum of the two numbers randomly selected is prime.

Name: ____________________

Answer Column
1A
1B
1C
1D
1E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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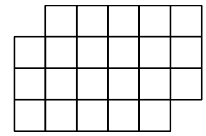
1D A 4″× 6″ rectangle is partitioned into 1″× 1″ squares. Then two opposite 1″× 1″corner squares are removed resulting in the figure at the right. How many squares with integer length sides can be found using the line segments in the resulting figure?



1E A sports jacket is on sale for 30% off the regular price. After the sale, the same sports jacket is marked up k% to again sell at the original regular price. Find k to the nearest whole number.

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Registered School: Leota Middle School (WOODINVILLE WA)

Division **M**

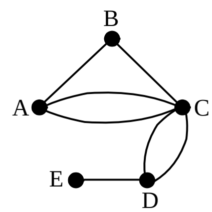
Mathematical Olympiads
December 11, 2019
For Elementary & Middle Schools

Contest **2**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

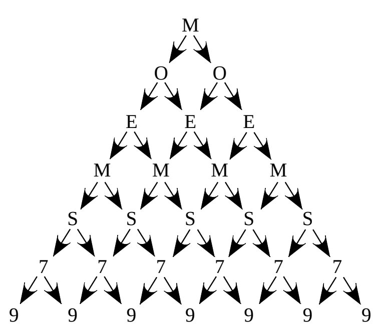
2A Teams A, B, C, D, and E are five local little league baseball teams. Each edge connecting two teams means that those teams have played a game against one another so far this season. Some teams have played each other twice, such as A and C. How many more games were played by the team with the greatest number of games played, than the team with the fewest number of games played?



2B The prime factorization of $2020 = 2^2 \times 5 \times 101$.

If $N = 2^A \times 5^B \times 101^C$, find the least value of $A + B + C$, so that $2020 \times N$ is a perfect cube.

2C MOEMS was founded in 1979 and is celebrating its 41st anniversary. Beginning with the top letter "M", following an arrow-directed path from top to bottom, will spell "MOEMS79". How many different top-to-bottom paths spell "MOEMS79"?



Name: ______________________________

Answer Column
2A
2B
2C
2D
2E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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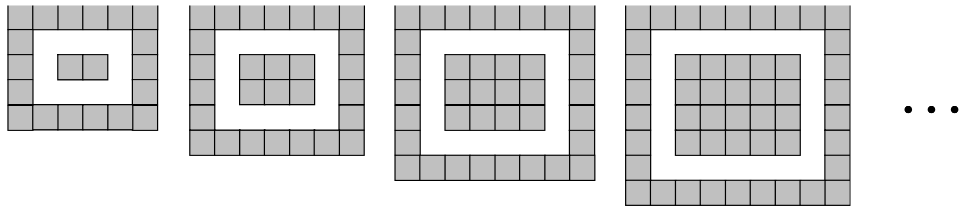
2D Each figure in the sequence is composed entirely of 1×1 shaded squares. If the pattern is continued, how many 1×1 shaded squares will appear in the 8^{th} figure?



2E Different letters stand for different digits, and no leading digit can equal 0, in the cryptarithm shown. Find the greatest possible value of "FOUR".

$$\begin{array}{r} \text{ONE} \\ \text{ONE} \\ \text{ONE} \\ +\ \text{ONE} \\ \hline \text{FOUR} \end{array}$$

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Division **M**

Mathematical Olympiads
January 15, 2020
For Elementary & Middle Schools

Contest **3**

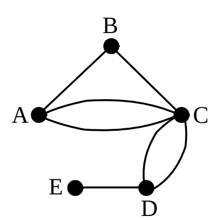
Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

3A Find the sum: $\frac{8}{4}+\frac{8}{0.4}+\frac{0.8}{0.4}+\frac{0.8}{4}$ as a decimal *to the nearest tenth.*

3B The speed of light is approximately 186,000 miles/second. If $186{,}000 = 2^{A} \times 3^{B} \times 5^{C} \times 31^{D}$, compute the simplified value of $(A + B) \times (C + D)$.

3C Teams A, B, C, D, and E are five local little league baseball teams. An edge connecting two teams means that those teams have played a game against one another so far this season. How many more games must be played so that each team will have played each of the other teams exactly twice?



Name: ____

Answer Column
3A
3B
3C
3D
3E
Do Not Write in this Space. *For PICO's Use Only.* *SCORE:*

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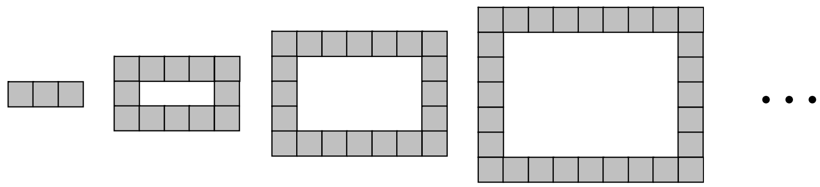
3D Each figure in the sequence is composed entirely of 1×1 shaded squares. If the pattern is continued, how many total 1×1 shaded squares will appear altogether in the first 50 figures?



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3E Different letters stand for different digits, and no leading digit can equal 0, in the cryptarithm shown. If each of the digits in the number SIX is even, find the value of the 3-digit number SIX.

$$\begin{array}{r} \text{TWO} \\ \text{TWO} \\ +\ \text{TWO} \\ \hline \text{S I X} \end{array}$$

Division **M**

Mathematical Olympiads
February 12, 2020
For Elementary & Middle Schools

Contest **4**

Directions to Students: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

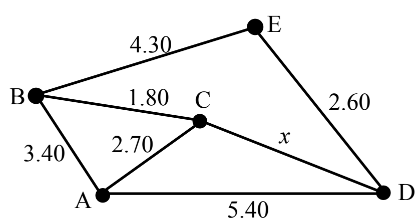
Name: ____

4A Compute the sum of all the numbers found in the figure.

		8			
	8	8	8		
8	8	11	11	11	3
	8	11	11	3	3
		11	3	3	3
		3	3	3	3

4B Define the symbol $[\![x]\!]$ to be the greatest integer less than or equal to x.
For example: $[\![12.7]\!]=12$, $[\![2]\!]=2$ and $[\![-3.2]\!]=-4$.
Compute the integer value of the expression: $\dfrac{[\![6.8]\!]\times[\![4.99]\!]}{[\![-2.1]\!]}$.

4C A network of toll roads and the cost (in dollars and cents) to travel between cities A, B, C, D, and E is illustrated. Find the greatest possible toll value for x, in dollars and cents, with 10 cent increments, so that the combined tolls from city A to city E is less than the cost of any other path from A to E.



Name: ____________________

Answer Column
4A
4B
4C
4D
4E
Do Not Write in this Space. For PICO's Use Only. *SCORE:*

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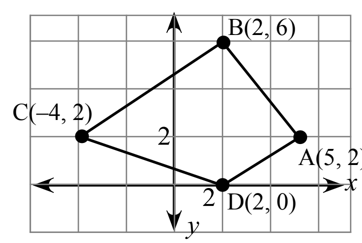
4D A, B, and C are distinct non-zero digits used to form the two 3-digit whole numbers ABC and CBA, so that ABC − CBA = 396. How many whole number values of ABC are possible?

4E Find the number of square units in the area of trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$, and vertices $A(5, 2)$, $B(2, 6)$, $C(-4, 2)$, and $D(2, 0)$ as illustrated.



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Division **M**

Mathematical Olympiads
March 11, 2020
For Elementary & Middle Schools

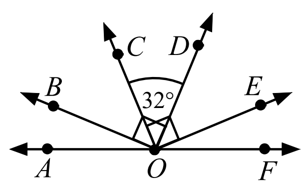
Contest **5**

<u>Directions to Students</u>: After all questions have been read by your PICO, you will have 30 minutes to complete this contest. You may not have a pen or pencil in your hand while the PICO reads the set of questions to the class. Calculators are not permitted. All work is to be done on the pages provided. No additional scrap paper is to be used. Answers must be placed in the corresponding boxes in the answer column.

Name: ____

5A Find the sum of all the integers from –7 to +5, including –7 and +5.

5B Straight line $\overleftrightarrow{AOF}$ and rays $\overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ and $\overrightarrow{OE}$ appear as shown. Additionally, $\angle BOD$ and $\angle COE$ are right angles, $m\angle COD = 32°$ and $\angle AOB \cong \angle EOF$. Find the number of degrees in the measure of $\angle AOB$.



5C Each of the digits 1, 2, 3, 4, 5, 6, and 7 is placed, one to a box, to form the two 2-digit numbers and one 3-digit number in the arithmetic problem shown. Find the greatest value for the number M if

$$\text{M} = \square\square\square - (\square\square - \square\square)$$

Name: ____

Answer Column
5A
5B
5C
5D
5E
Do Not Write in this Space. *For PICO's Use Only.* *SCORE:*

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5D MOEMS was created in 1979.
For all values of a and b: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Compute the simplified numeric value of the expression: $\dfrac{(1900+79)^3 - 1900^3 - 79^3}{1900 \times 79 \times 1979}$.

5E A and B are two positive integers which differ by 12. How many values are possible for the greatest common factor of A and B?

Division **M**

Mathematical Olympiads
November 13, 2019
For Elementary & Middle Schools

Contest **1**

SOLUTIONS AND ANSWERS

1A	505
1B	3
1C	$\frac{18}{25}$ or equivalent
1D	42
1E	43

1A <u>Strategy</u>: *Use prime factorization.*
In order for a number to be a perfect square, each prime factor needs to appear an even number of times. Since $2020 = 2^2 \times 5 \times 101$, we need another 5 and another 101. Therefore, N = 5 × 101 = **505**.

FOLLOW-UP: Find the least value of N so that 3030 × N is a perfect square. [3030]

1B **METHOD 1** <u>Strategy</u>: *Create a net for a cube.*
Three faces meet at each vertex, so we need a minimum of **3** colors.

	1		
3	2	3	2
	1		

METHOD 2 <u>Strategy</u>: *Notice that each face is adjacent to four other faces.*
If we place a color on one face, then that same color cannot be on any adjacent face. The same color can only be placed on an opposite face. There are 3 pairs of faces.

FOLLOW-UP: What is the minimum number of colors needed to color an octahedron if adjacent faces must have different colors? [2]

1C **METHOD 1** <u>Strategy</u>: *Make a list or table of all possible sums.*
List all of the possible sums.

2 + 1 = 3	4 + 1 = 5	6 + 1 = 7	8 + 1 = 9	10 + 1 = 11
2 + 3 = 5	4 + 3 = 7	6 + 3 = 9	8 + 3 = 11	10 + 3 = 13
2 + 5 = 7	4 + 5 = 9	6 + 5 = 11	8 + 5 = 13	10 + 5 = 15
2 + 7 = 9	4 + 7 = 11	6 + 7 = 13	8 + 7 = 15	10 + 7 = 17
2 + 9 = 11	4 + 9 = 13	6 + 9 = 15	8 + 9 = 17	10 + 9 = 19

There are 25 possible sums and 18 of them result in a prime number so the probability is $\mathbf{\frac{18}{25}}$.

METHOD 2 <u>Strategy</u>: *Determine the possible prime number sums.*
The only prime number sums that can be created using a number from each set are: 3, 5, 7, 11, 13, 17, and 19. There is 1 way to sum to 3, 2 ways to sum to 5, 3 ways to sum to 7, 5 ways to sum to 11, 4 ways to sum to 13, 2 ways to sum to 17, and 1 way to sum to 19. The total number of ways to sum to a prime is 1 + 2 + 3 + 5 + 4 + 2 + 1 = 18. There are 5 × 5 = 25 possible pairs; therefore the probability is 18/25.

FOLLOW-UP: In the original problem, what is the probability that a product of the two numbers selected will be a prime number? [1/25]

1D **METHOD 1** *<u>Strategy</u>: Count the number of different sized squares.*
There are 22 squares that are 1×1.
There are 13 squares that are 2×2.
There are 6 squares that are 3×3.
There is 1 square that is 4×4.
Thus, there is a total of **42** squares.

METHOD 2 *<u>Strategy</u>: Examine the pattern of squares if the two corner squares were not removed.*
There would be $4 \times 6 = 24$ squares 1×1.
There would be $3 \times 5 = 15$ squares 2×2.
There would be $2 \times 4 = 8$ squares 3×3.
There would be $1 \times 3 = 3$ squares 4×4.
There is a total of $24 + 15 + 8 + 3 = 50$ squares. We need to remove any squares that contain either of the two removed corners. For each of the sized square listed above, there are two that contain an eliminated corner square. Therefore, there are $50 - 2 \times 4 = 42$ squares.

FOLLOW-UP: How many rectangles are in the given diagram that have an area of 6? [20]

1E **METHOD 1** *<u>Strategy</u>: Calculate the price in terms of k.*
If the jacket is on sale for 30% off, the item sells for 70% of the original price. The item is then marked up $k\%$ in order to restore the original price. Let p = the original price. Then, $0.7p + (0.7p)k\% = p$ so $0.7pk\% = 0.3p$ and $k\% = \frac{k}{100} = \frac{0.3p}{0.7p}$. It follows that $k \approx 42.857 =$ **43** to the nearest whole number.

METHOD 2 *<u>Strategy</u>: Choose a convenient price for the jacket.*
Let the jacket cost \$100. Then the sale price is \$70. To restore the price to \$100, add \$30. This is 30/70 or approximately 43%. Therefore, $k = 43$.

METHOD 3 *<u>Strategy</u>: Solve using reciprocal fractions.*
A discount of 30% means the jacket now sells for 70% of the original price; this is 7/10 of the original price. To restore the price to the original, multiply the discounted price by 10/7. This is 3/7 more than the discounted price or $42.857\% \approx 43\%$. Thus, $k = 43$.

FOLLOW-UP: An item in the store is discounted 40%. The item is then placed on sale for 75% off of the lower price. By what percent is the item reduced after these two discounts are applied? [85%]

NOTE: *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in "Creative Problem Solving in School Mathematics."*
Visit <u>www.moems.org</u> for details and to order.

SOLUTIONS AND ANSWERS

2A	4
2B	5
2C	64
2D	118
2E	3940

2A **METHOD 1** *Strategy: Count the number of edges that meet at each vertex.*
The number of games played by each team is equal to the number of edges that meet at each vertex. Therefore, Team A played 3 games, B played 2, C played 5, D played 3, and E played 1 game. The difference between the greatest number of games and the least number of games is 5 – 1 = **4**.

METHOD 2 *Strategy: Calculate the number of games played by each team.*

Team	A	B	C	D	E
Games Played	B once C twice	A once C once	A twice B once D twice	C twice E once	D once
Total Games	3	2	5 (most)	3	1 (least)

The team playing the most games – the team playing the least = 5 – 1 = 4.

FOLLOW-UP: How many games must be played if each team plays each of the other teams exactly once? [10]

2B **METHOD 1** *Strategy: Apply the definition of a perfect cube.*
A number is a perfect cube when its prime factors each occur a multiple of three times. Since $2020 = 2^2 \times 5 \times 101$ and we wish to make $2020 \times N$ be the least possible perfect cube where $N = 2^A \times 5^B \times 101^C$, $2020 \times N = 2^3 \times 5^3 \times 101^3$. Use the product rule for exponents [$a^m \times a^n = a^{m+n}$]. Thus A = 1, B = 2, and C = 2 so the least possible sum is A + B + C = 1 + 2 + 2 = **5**.

METHOD 2 *Strategy: Determine the smallest such perfect cube.*
$(2 \times 5 \times 101) = 1010^1$
$(2 \times 5 \times 101)^3 = (2 \times 5 \times 101)(2 \times 5 \times 101)(2 \times 5 \times 101)$
$(2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 101 \times 101 \times 101) = 1010^3$
We need one more 2 (A = 1), two more 5's (B = 2), and two more 101's (C = 2).
With these we have $(2020) \times (2^1 \times 5^2 \times 101^2) = 1010^3$ which is a perfect cube.
Finally, A + B + C = 5.

FOLLOW-UP: Integers to the fourth power are called "tesseractic numbers." What is least possible value of N if 2020 × N is a tesseractic number? [515,150,500]

2C **METHOD 1** *Strategy: Start at the top and consider the number of paths.*
From the top M, there are 2 paths to get to the O in row 2, moving left (L) or right (R). From the top M, there are 4 paths to get to the E in row 3 (LL, LR, RL, RR). From the top M, there are 8 paths to get to the M in row 4 (LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR). Notice how each level adds a factor of 2. Therefore, the S in row 5 would yield 16 paths, the 7 in row 6 would yield 32 paths, and the 9 in row 6 would yield **64** paths.

METHOD 2 <u>Strategy</u>: *Use Pascal's Triangle. (A fertile area to investigate!)*
Count the number of paths that lead to each of the symbols in the triangle. You will discover the numbers in the triangle at the right. Summing each row gives the number of ways to get to the row starting at the top.
Since 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64, there are 64 paths from the letter "M" to the number 9.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

FOLLOW-UP: Yummy Yogurt offers the following toppings: sprinkles, marshmallows, walnuts, Reese's Pieces, M & M's, hot fudge, and whipped cream. How many topping combinations are possible? [128]

2D **METHOD 1** <u>Strategy</u>: *Count the number of squares and look for a pattern.*

Figure	1	2	3	4	. . .	8
Dimensions	5×6	6×7	7×8	8×9	. . .	12×13
1×1 squares	2(5) + 2(4) + 2	2(6) + 2(5) + 6	2(7) + 2(6) + 12	2(8) + 2(7) + 20	. . .	2(12) + 2(11) + 72

Therefore, there are 2(12) + 2(11) + 72 = **118** small 1×1 squares in the eighth figure.

METHOD 2 <u>Strategy</u>: *Use subtraction and addition of areas.*
Subtract the area of the middle rectangle from the area of the outer rectangle and then add back the area of the innermost rectangle.
The shaded area of the first figure is $(6 \times 5) - (4 \times 3) + (2 \times 1) = 30 - 12 + 2 = 20$.
The shaded area of the second figure is $(7 \times 6) - (5 \times 4) + (3 \times 2) = 42 - 20 + 6 = 28$.
The shaded area of the third figure is $(8 \times 7) - (6 \times 5) + (4 \times 3) = 56 - 30 + 12 = 38$.
The shaded area of the fourth figure is $(9 \times 8) - (7 \times 6) + (5 \times 4) = 72 - 42 + 20 = 50$.
The shaded area of the eighth figure is $(13 \times 12) - (11 \times 10) + (9 \times 8) = 156 - 110 + 72 = 118$.

FOLLOW-UP: In the 5th figure, what is the ratio of the area of the inner shaded rectangle to the area of the outermost rectangle? [1/3]

2E **METHOD 1** <u>Strategy</u>: *Use trial and error to eliminate possible values for "ONE".*
Start with the greatest possible 3-digit number that satisfies the criteria: 987 + 987 + 987 + 987 = 3948. You would eliminate this choice when you add the E's and notice R = 8, the value of N. The next greatest possible number, $986 \times 4 = 3944$. This is eliminated since U = R. The next is $985 \times 4 =$ **3940.**

METHOD 2 <u>Strategy</u>: *Work from left to right.*
The value of F is formed by regrouping the sum of the four O's. The largest possible value for F is 3 and occurs when O is either 8 or 9. Try 9 for O. Four 9's equals 36 with 6 in the hundreds place and the regrouping result of 3 for F. But the sum of the O's must also be O, meaning that the four N's need to regroup to form a 3 as well. This leads to F replaced by 3, O by 9, and N by 8. Replacing E by 7, results in both R and N being 8, which does not satisfy the conditions. Replace E with 6. This results in both U and R being 4, which does not satisfy the conditions. Replace E with 5, the next largest value. This results in U being 4 and R being 0. This works and results in "FOUR" equal to 3940.

FOLLOW-UP: Use the same cryptorithm to find the least possible sum. [1304]

NOTE: *Other FOLLOW-UP problems related to some of the above can be found in our four contest problem books and in "Creative Problem Solving in School Mathematics."*
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SOLUTIONS AND ANSWERS

3A	24.2
3B	20
3C	13
3D	9999
3E	408

3A **METHOD 1** *Strategy: Write each fraction as a decimal and add.*

Convert as follows: 8/4 = 2; 8/0.4 = 80/4 = 20; 0.8/0.4 = 8/4 = 2; 0.8/4 = 0.2. Then the sum is 2 + 20 + 2 + 0.2 = **24.2**

METHOD 2 *Strategy: Find a common denominator, add, then simplify.*
The common denominator is 4 so 8/4 + 80/4 +8/4 + 0.8/4 = 96.8/ 4 = 24.2.

FOLLOW-UPS: (1) Find the sum: 4/8 +0.4/8 + 0.4/0.8 + 4/0.8 [6.05]
(2) Is the sum of the reciprocals equal to the reciprocal of the sum? [No]

3B **METHOD 1** *Strategy: Factor 186,000 into simpler numbers and continue to factor until all factors are primes. Below is one example of such a factoring.*

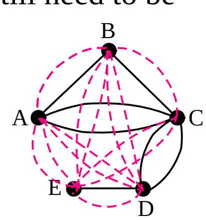
$$186{,}000 = 6 \times 31 \times 10^3$$
$$= 2 \times 3 \times 31 \times (2 \times 5)^3$$
$$= 2 \times 3 \times 31 \times 2^3 \times 5^3 = 2^4 \times 3^1 \times 5^3 \times 31^1.$$

Thus A = 4, B = 1, C = 3, D = 1 and (A + B)×(C + D) = (4 + 1)(3 + 1) = (5)(4) = **20**.

METHOD 2 *Strategy: Continuously divide by possible prime factors.*
Since 186,000 is even, $186{,}000 = 93{,}000 \times 2 = (46{,}500 \times 2) \times 2 =$ $(23{,}250 \times 2) \times 2 \times 2 = (11{,}625 \times 2) \times 2 \times 2 \times 2 = (2{,}325 \times 5) \times 2^4 =$ $(465 \times 5) \times 5 \times 2^4 = (93 \times 5) \times 5 \times 5 \times 2^4 = (31 \times 3) \times 5^3 \times 2^4$. Then, $186{,}000 = 2^4 \times 3^1 \times 5^3 \times 31^1$ and $(A + B) \times (C + D) = (4+1) \times (3+1) = (5)(4) = 20$.

FOLLOW-UP: There are approximately 1,600 meters in a mile. What is the speed of light (approximately 186,000 miles/second) in meters/second to the nearest 100,000,000 meters/second? [3×10^8]

3C **METHOD 1** *Strategy: Subtract the number of games played from the total number of games played when each team plays each of the other teams twice.*
Each of the 5 teams can play each of the remaining 4 teams in 5 × 4 = 20 ways, which would include A playing B and B playing A. Since there are 7 edges in the diagram, there were 7 games played. The number of games that still need to be played is 20 – 7 = **13**.



METHOD 2 *Strategy: Draw additional lines.*
Draw and count the number of additional lines needed so that there will be 2 lines connecting each pair of teams.

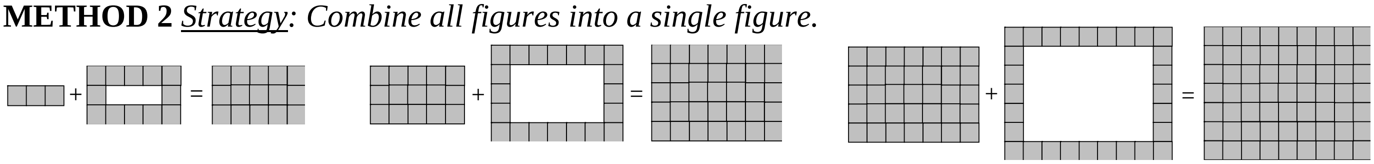
FOLLOW-UPS: (1) How many additional games would be needed if each team played each of the others 3 times? [23] *(2) How many additional games would be needed if a 6th team was added to the original problem?* [23] *(3) The NCAA March Madness begins with 64 teams. In each game the loser is eliminated, and the winner plays again. How many games must be played to determine a champion?* [63]

3D **METHOD 1** <u>Strategy</u>*: Find each successive sum and look for a pattern.*

Figure	1	2	3	4	. . .	50
# of squares	3	12	20	28	. . .	
Total	3	3 + 12 = 15	15 + 20 = 35	35 + 28 = 63	. . .	
Pattern	1 × 3 = 3	3 × 5 = 15	5 × 7 = 35	7 × 9 = 63	. . .	99 × 101 = 9999

Notice that the first factor in the pattern, 1, 3, 5, and 7 is 1 less than twice the number of the figure. The second factor is 2 more than the first or 1 more than twice the number of the figure. The sum of the first 50 figures would be the product of 2 × 50 – 1 and 2 × 50 + 1, or 99 × 101 = **9999.**

METHOD 2 <u>Strategy</u>*: Combine all figures into a single figure.*



Figures 1 + 2 Figures 1 + 2 + 3 Figures 1 + 2 + 3 + 4

Notice that the sum of the first 2 figures results in a filled 3 × 5 rectangle. The sum of the first 3 figures results in a filled 5 × 7 rectangle. The sum of the first 4 figures results in a filled rectangle that is 7 × 9. In general, the sum of the first n figures results in a $(2n - 1) \times (2n + 1)$ rectangle. Therefore, the sum of the first 50 figures will be the area of a filled rectangle with an area of 99 × 101 = 9999.

FOLLOW-UPS: (1) Find the sum of the first 100 odd positive integers. [10,000] *(2) Find the sum of the first 100 even positive integers.* [10,100] *(3) Find the sum of the first 100 positive integers.* [5050]

3E **METHOD 1** <u>Strategy</u>*: Use numerical reasoning along with guess and check*

Since S, I, and X must be even digits, the letter O must be even; but not 0. In addition, since T + T + T results in a single digit, T = 1 or 2. Let T = 1 and O = 2. If O = 2, then X = 6. The letter W must be even since there is no previous regrouping and large enough to force a 1 to be added to the hundred's digit. If W = 4, then three times 4 is 12, and I = 2 = O, which does not satisfy the condition that each digit is unique. W can't be 6 since X is 6. The letter W can't be 8 since then a 2 would be carried to the hundred's place, making S odd. Repeat the process if O = 4; then X = 2 and 1 is brought into the ten's place. The letter W must be odd. It can't be 1 (T = 1), so try 3. We have TWO = 134 and SIX = 406, but O cannot equal S. If W = 5, TWO = 154 and SIX = 462, same problem. The letter W cannot be 7 (carry a 2) if T = 1.
Let O = 6 and W = 3. We then have TWO = 136 and SIX = **408**, which satisfies the condition that the digits are unique and S, I, and X are even.

METHOD 2 <u>Strategy</u>*: Apply the process of elimination.*

T must be 1 or 2 (see method 1). If T = 2 and then S = 6 there are 2 possible values for TWO: 2W4 and 2W8. Neither is possible since 3 × 2W4 → X = 2 and 3 × 2W8 → X = 4 and W must also be even. If T = 2 and S = 8 there are 2 possible values for TWO: 2W4 and 2W6. Neither is possible since 3 × 2W4 → X = 2 and 3 × 2W6 → X = 8. Thus T = 1 and S = 4. The three possible values for TWO are 1W2, 1W6, and 1W8. Multiply each by 3 to find that X = 6, 8, or 4 respectively. The list of possible values for TWO is now 102, 182, 136, 156, 176, 196, and 108. Test to see that 136 is the only value that produces even digits for SIX without duplication. When TWO = 136, SIX = 408.

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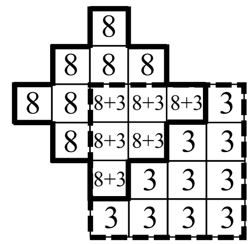
Visit <u>*www.moems.org*</u> *for details and to order.*

Division M | Mathematical Olympiads February 12, 2020 For Elementary & Middle Schools | Contest 4

SOLUTIONS AND ANSWERS

4A **METHOD 1** <u>Strategy</u>: *Recognize that the 11s are 8s and 3s.*
Notice that the 11 = 8 + 3. The figure is really a "star" of 8s and a square of 3s that overlap (the overlap is the 11s). There are sixteen (4 × 4 = 16) 3s which sum to 16 × 3 = 48. There are thirteen (1 + 3 + 5 + 3 + 1 = 13) 8s which sum to 13 × 8 = 104. The sum of all the numbers found in all of the squares is 48 + 104 = **152**.



METHOD 2 <u>Strategy</u>: *Add each row.*
The sums of the rows are 8, 24, 52, 36, 20, and 12. These add to 152.

METHOD 3 <u>Strategy</u>: *Add the 8s, the 11s and the 3s.*
There are seven 8s, six 11s, and ten 3s and 7 × 8 + 6 × 11 + 10 × 3 = 152.

FOLLOW-UP: Find the sum of all the numbers in the trapezoidal shape. [360]

5 5 5 5 5 12 7 7 7 7 7
5 5 5 5 12 12 7 7 7 7
5 5 5 12 12 12 7 7 7
5 5 12 12 12 12 7 7
5 12 12 12 12 12 7

4B <u>Strategy</u>: *Use the given definition and order of operations.*
First, use the definition of $[\![x]\!]$ to obtain $[\![6.8]\!]=6$, $[\![4.99]\!]=4$, and $[\![-2.1]\!]=-3$.

Then evaluate $\frac{6\times4}{-3}=$ **–8**.

FOLLOW- UP: If the symbol $\lfloor x \rfloor$ means the least integer greater than or equal to x,

compute the value of $\frac{\lfloor 6.8\rfloor\times\lfloor 4.99\rfloor}{\lfloor -2.1\rfloor}$. [–17.5]

4C <u>Strategy</u>: *Use a list to organize the possible paths from A to E.*
Compare the costs of the various paths.
The path ABE costs \$3.40 + \$4.30 = \$7.70.
The path ABCDE is more expensive than ABE.
The path ACBE costs \$2.70 + \$1.80 + \$4.30 = \$8.80 which is more than ABE.
The path ACDE costs \$2.70 + \$$x$ + \$2.60 = \$5.30 + x.
The path ADCBE is more than \$7.70, as is the path ADE.
If the path ACDE is less than \$7.70, and x is as large as possible, and the cost must be a multiple of \$0.10, the path ACDE costs \$7.60 and the value of x is **\$2.30**.

FOLLOW-UP: If the cost of the path from C to D was \$2.30 and a path could be offered from C to E, what is the greatest price, with \$0.10 increments, that could be offered to make the trip less than ACDE? [\$4.80]

4A	**152**
4B	**–8**
4C	**\$2.30**
4D	**35**
4E	**27**

4D **METHOD 1** *<u>Strategy</u>: Apply number sense and algebra to find possible values.*
Since ABC – CBA = 396, C – A = 6 or C + 10 – A = 6. The difference of the numbers is a positive number so A > C and therefore, C + 10 – A = 6 → A = C + 4. It follows that A can equal 5, 6, 7, 8, or 9. For each of these values there is only one possible value for C: 1, 2, 3, 4, or 5 respectively. For each pair of values for A and C there are 7 possible values B since all three values are distinct. Thus, there are $5 \times 7 =$ **35** possible values for ABC.

METHOD 2 *<u>Strategy</u>: Convert subtraction into addition and then create a table.*
Rewrite the problem as ABC = CBA + 396. Since C < A, A + 6 = C + 10 so A = C + 4. Notice that the value of B is any non-zero digit since the 3-digit number CB(C + 4) + 396 = (C + 4)BC for all B values.

C	1	2	3	4	5
A	5	6	7	8	9

There are 5 pairs of numbers for A and C, and B can be any digit other than A, C or 0. Therefore, there are $5 \times 7 = 35$ positive integers that satisfy the conditions for ABC.

FOLLOW-UP: A, B, C and D are distinct non-zero digits used to form the two 4-digit whole numbers ABCD and DCBA, so that ABCD − DCBA = 1089. How many whole number values of ABCD are possible? [42]

4E **METHOD 1** *<u>Strategy</u>: Use the Area of a triangle formula.*
Connect A to C. The base of ΔABC is AC = 5 – (–4) = 9 so its area = (1/2)(9)(4) = 18.
The base of ΔADC is AC = 5 – (–4) = 9 so its area = (1/2)(9)(2) = 9.
The total area of the trapezoid = 18 + 9 = **27** square units.

METHOD 2 *<u>Strategy</u>: Draw a rectangle that contains all 4 vertices of the trapezoid.*
The vertices of the rectangle would be E(–4, 6), F(5, 6), G(5, 0), and H(–4, 0). The area of rectangle EFGH is $9 \times 6 = 54$. Subtract the areas of the 4 right triangles (ΔEBC, ΔFAB, ΔGAD, and ΔHDC) inside the rectangle but outside the trapezoid: $A = 54 - \left(\frac{6\times4}{2}+\frac{3\times4}{2}+\frac{3\times2}{2}+\frac{6\times2}{2}\right) = 54 - (12 + 6 + 3 + 6) = 27.$

METHOD 3 *<u>Strategy</u>: Draw the two diagonals to divide the shape into 4 right triangles.*
Label the intersection of the diagonals O. The area of ABCD is the sum of the areas of the 4 right triangles: $A_{\Delta OAB} = (4 \times 3)/2 = 6$, $A_{\Delta OAD} = (3 \times 2)/2 = 3$, $A_{\Delta OCD} = (6 \times 2)/2 = 6$, $A_{\Delta OBC} = (6 \times 4)/2 = 12$. The area of the trapezoid is 6 + 3 + 6 + 12 = 27.

METHOD 4 *<u>Strategy</u>: Use the Shoelace Theorem.*
The Shoelace Theorem can be used to find the area of a polygon given its coordinates. The coordinates of the vertices of the trapezoid are: (5, 2), (2, 6), (–4, 2), and (2, 0). The theorem states that the area of the trapezoid equals:

$$\frac{\left|\left((5)(6)-(2)(2)\right)+\left((2)(2)-(6)(-4)\right)+\left((-4)(0)-(2)(2)\right)+\left((2)(2)-(0)(5)\right)\right|}{2}=\frac{|26+28-4+4|}{2}=27.$$

FOLLOW-UPS: (1) In the original problem, add 2 to every positive coordinate and subtract 2 from every negative coordinate. Find the area of the resulting quadrilateral. [52] *(2) Quadrilateral TECH has vertices T(0,6), E(8,0), C(–4,–6), and H(–12,0). Find the area of TECH.* [120]

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Division **M**

Mathematical Olympiads
March 11, 2020
For Elementary & Middle Schools

Contest **5**

SOLUTIONS AND ANSWERS

5A **METHOD 1** Strategy: *Use pairs that sum to zero.*
Adding all the integers from –7 to +5, both –5 and +5, –4 and +4, –3 and +3, etc., occur. Since these pairs sum to zero, only –7 and –6 remain. Thus, the sum is **–13**.

METHOD 2 Strategy: *Separate negative and positive integers, then combine.*
$-7 + -6 + -5 + -4 + -3 + -2 + -1 = -28$.
0 + 1 + 2 + 3 + 4 + 5 = 15
–28 + 15 = –13.

Follow-ups: (1) Find the sum of all even integers from −14 to +10. [–26]
(2) Find 3 consecutive integers whose sum is –15. [–6, –5, – 4]

5B **METHOD 1** Strategy: *Use the fact that there are 180° in a straight angle (line).*
Determine the number of degrees in $\angle BOE$: $m\angle BOE = 90° + 90° - 32° = 148°$.
Since $\angle AOB \cong \angle EOF$, we have $2 \times m\angle AOB + 148° = 180°$. Thus, $m\angle AOB =$ **16°**.

METHOD 2 Strategy: *Use algebra.*
Since $m\angle BOD = m\angle COE = 90°$, $m\angle BOC = m\angle EOD = 90° - 32° = 58°$. Let $x = m\angle AOB = m\angle EOF$. Solve for x: $x + 58° + 32° + 58° + x = 180° \to x = 16°$.

Follow-ups: (1) If $m\angle COD = (10n)°$, find the number of degrees in $\angle AOB$ in terms of n. [5n°] *(2) If one base angle of an isosceles triangle has a measure of 15°, what is the measure of the vertex angle?* [150°]

5C **METHOD 1** Strategy: *Make the first number as great as possible while making the second (in the parentheses) the least value possible.*
The largest the first number can be is 765. That leaves the digits 1, 2, 3, and 4. We want the difference of the two numbers in the parentheses to be as minimal as possible. Since negative numbers are less than positive numbers, we want the difference to be a negative number with the greatest possible absolute value. That difference is 12 – 43 = –31 and 765 – (–31) = 796. This, however, is not the greatest possible value. If we switch the position of the 5 and 4, we create a greater absolute value difference in the parentheses, so the final answer is 764 – (12 – 53) = **805**.

METHOD 2 Strategy: *Use algebra.*
Rewrite the problem as $100a + 10b + c - [(10d + e) - (10f + g)]$. Factor out the common factors: $100a + 10(b - d + f) + (c - e + g)$. Now we see that $a = 7$, $b + f - d$ is greatest when b and f are 6 and 5 in either order and d is 1. Finally, c and g should be 4 and 3 while $e = 2$. Therefore, there are several possible arrangements that produce the greatest value of 805.

Follow-ups: (1) What is the least value of M? [83]
(2) What is the greatest value of M if M = □□□ – (□□ + □□)? [728]

5A	**–13**
5B	**16**
5C	**805**
5D	**3**
5E	**6**

5D **METHOD 1** *Strategy: Apply the given formula to expand the cubed expression.*
By formula: $(1900 + 79)^3 = (1900)^3 + 3 \times (1900^2 \times 79) + 3 \times (1900 \times 79^2) + (79)^3$. Combine the terms in the numerator to get $3 \times 1900^2 \times 79 + 3 \times 1900 \times 79^2 = 3 \times 1900 \times 79 \times (1900 + 79)$. Use the property that $\frac{a}{a} = 1$ to simplify the fraction: $\frac{3\times1900\times79\times(1900+79)}{1900\times79\times1979} = 3\times\frac{1900}{1900}\times\frac{79}{79}\times\frac{1900+79}{1979} = \mathbf{3}$.

METHOD 2 *Strategy: Assign variables to each of the numbers.*

Let $a = 1900$ and $b = 79$. Then, $a + b = 1979$. Rewrite the question using variables: $\frac{(a+b)^3 - a^3 - b^3}{ab(a+b)}$.

Replace $(a + b)^3$: $\frac{(a+b)^3 - a^3 - b^3}{ab(a+b)} = \frac{(a^3 + 3a^2b + 3ab^2 + b^3) - a^3 - b^3}{ab(a+b)} = \frac{3a^2b + 3ab^2}{ab(a+b)} = \frac{3ab(a+b)}{ab(a+b)} = 3$.
Notice the result is equal to 3 no matter what the values are for a and b.

FOLLOW-UP: Use the formulas $(a + b)(a + b) = a^2 + 2ab + b^2$ and $(a + b)(a - b) = a^2 - b^2$ to calculate each of the following: (1) 2020^2 and (2) 2003×1997. [4,080,400 and 3,999,991]

5E **METHOD 1** *Strategy: Use the concept that if x is a factor of A and B, then x is a factor of A − B.*
Since A − B = 12, then if x is a factor of A and x is a factor of B (we are looking for common factors of both), then x is necessarily a factor of 12, which has 6 factors: 1, 2, 3, 4, 6, and 12. Any one of these may be the greatest common factor of A and B. Thus, there are **6** possible GCF values.

METHOD 2 *Strategy: Use guess and check.*
Find pairs of numbers that differ by 12 and examine their greatest common factors (GCF): 1 and 13 differ by 12 and have a GCF of 1. Other examples are: GCF(2, 14) = 2, GCF(3, 15) = 3, GCF(4, 16) = 4, GCF(5, 17) = 1, GCF(6, 18) = 6 etc. Notice that the GCF for each pair of numbers are factors of 12. The only remaining factor is 12 and the GCF(12, 24) = 12. There are 6 values possible for the GCF.

FOLLOW-UP: What is the greatest common factor for each pair of numbers which also differ by 12: (203, 215), (200, 212), and (600, 612)? [1, 4, and 12]

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